Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

CRITICISMS AND DISCUSSIONS.

LOGIC AS RELATION-LORE.

REPLY TO MR. FRANCIS C. RUSSELL.

Mr. Francis C. Russell has devoted several pages in *The Monist*, Vol. III, No. 2, to discussing with evident sympathy the reflexions which in 1891 I presented to the readers of the French *Revue Philosophique*, concerning the theory of relations and its application to the concept of mathematical equality.

While approving the spirit that has inspired me, Mr. Russell has nevertheless formulated a large number of criticisms. Indeed, he accepts scarcely any of the data of my work, and he wholly rejects the results thereof. If these criticisms found their origin in a point of view different from mine, I should certainly abstain from making any reply thereto. Any controversy upon any special subject, when the parties thereto differ already as to the manner of treating every subject, is wholly lacking in interest. In virtue, perhaps, of the principle of contradiction, it is only apt to confirm each one of the contestants as to the merits of his own theory. But such is not here the case. It appears to me that between Mr. Russell and myself there exists a certain community of view, which makes discussion possible and even profitable. Besides this, the observations of Mr. Russell have, I believe, their main occasion in certain misunderstandings, arising in part from my honorable opponent not having perhaps gained a sufficiently intimate acquaintance with my article (which would be very natural, since a review article is not one of those productions one is apt to look into very deeply), and, doubtless, also in part from a lack of precision and lucidity in the explanation that I have given of my ideas. I am thus justified, I think, in continuing the debate.

Without wishing, however, to enter into a complete discussion, one that would go to the bottom of the subject, and in order also not to abuse the hospitality which the good-will of Dr. Carus has accorded to me, I will content myself with correcting the misunderstandings, which I consider the results of erroneous interpretations of certain points of my article. This will be for me, moreover, an occasion for putting more in relief what I hold as fundamental in the theory of cognition, but in so far only as it relates to the cognition of the relations between things and to the exclusion of the theory of the cognition of the things themselves.

I. THE AXIOM OF SYMMETRY.

And in the very first place I ought to separate from the theory of cognition the questions relative to the validity of the Axiom of Symmetry.

After having remarked that every philosophic theory is founded upon some very general supposition, taken as established, Mr. Russell declares that the general supposition upon which the theory which I maintain is founded is that axiom relative to symmetric relations (I mean those having no determinate *sense*, or convertible without alteration), which I have enounced as follows :

"Two things which have the same symmetric relation to a third thing, have between them that same relation."

This understanding of Mr. Russell is most certainly unfounded, and I must disavow it. I have undoubtedly tendered the axiom in question as a fundamental one, that is to say, in respect of placing myself at the scientific point of view, at the point of view not of conceivability, but of reality. In this sense, the axiom is certainly one of the most general laws of exterior phenomena, but it does not follow that in logic its part is an essential one ; it is merely a rule of simplification. For instance, the axiom permits us to reduce certain complexes of relations that are known primarily as having three or more determinate terms to two determinate terms ; but should the axiom be in fault, the complexes of three terms or more would none the less preserve its conceptual value.

If, then, the axiom of symmetry is one of the objective laws that permits of abstraction, that is to say, that enables us to simplify our cognitions in so far as the same consist in the classification of exterior objects, it is in no manner or form one of the principles that govern the development of our cognitions. So well able would I have been to suppress all mention of it without doing any violence to the theory which I maintain, that it would have been sufficient for me in the concrete examples to substitute for the axiom of symmetry the special laws of each science ; laws which are denied by no one, no matter what rank is accorded them ; laws of which the axiom itself is only an abstract enunciation, common to all, but unable to overstep them.

After having thus wrongfully imputed to the axiom of symmetry a fundamental rôle in my theory, Mr. Russell contests its generality, from whence follows the natural conclusion, one, however, which he has not brought forward, that my theory is unsound.

It is plain that we cannot consider the generality of the axiom as established, so long as it has not been verified in every case of that generality. I am too much the adversary of *a priori* certitude and "natural verities" to dream of imputing to the axiom any transcendental virtue.

Still the examples that Mr. Russell has cited as impugning it, do not appear to me to be of a nature fit to confirm his opinion. I beg leave to discuss them briefly, in virtue besides of the purely scientific interest that attaches to the question

One of these examples is borrowed from the solar system. Mr. Russell holds that the planets ought to be considered as being in equilibrium with the sun, without, (as the axiom would have it and since equilibrium is a symmetric relation,) being in equilibrium with each other.

It is certainly a scientific error from the dynamic point of view, abstraction being made of the initial velocities, to consider the rôle played by the sun in the system composed of the sun and the planets to be different from the rôle played by the planets, the single distinctive peculiarity of the sun in the case being its relatively enormous mass, and, in consequence thereof, the closer proximity of its centre of gravity to the centre of gravity of the entire system.

But that is only a difference of degree, not of nature. In reality, the only symmetric relation which I perceive between the planets and the sun is not the relation of equilibrium, it is the relation of mutual attraction, such as was discovered and defined by Newton. So far from the planets being in equilibrium with the sun, they fall towards the sun each instant, abstraction being made of their initial velocities, and at each instant the sun falls towards the planets. But the same can be said of two planets, as related each to the other. There exists a mutual attraction between them, like that between each one of them and the sun. The example of the solar system does not impugn the axiom of symmetry, it is, on the contrary, a confirmation thereof ; perhaps the clearest and simplest.

The same may be said of the geometric example invoked by Mr. Russell, who convinced *a priori* of the futility of the axiom, has not taken good care to weigh all his objections. "The distance of points from each other," says Mr. Russell, "is a "symmetrical relation, and yet point *A* may be from point *C* the very same distance "that point *B* is from *C*, but the distance of points *A* and *B* from one another may "vary from coincidence to double the distance *AC*—*BC*."

It is easy to reply to this objection, which can be interpreted in two ways. If the nature of the symmetric relation that Mr. Russell contemplates is the equality of lengths, the axiom of symmetry is alien to the question, since it supposes two symmetric relations as given, while in the case made only a single relation of that kind is given.

If he refers simply to the relations of relative position, or to the coexistence of two points in space, the axiom is satisfied, because two points which coexist with a third, coexist with each other.

If, lastly, the two points *A* and *B* coincide, we are no longer in condition to apply the axiom, since the same requires three distinct terms.

There remains a third example, drawn from the ethical domain ; namely, "mutual friendship is certainly a symmetrical relation, but *A* and *C* may be mutual "friends, and *B* and *C* mutual friends also, but it in no wise follows that *A* and *B* "are friends."

This objection, at first blush, appears embarrassing. It will evidently not suffice to answer in the words of the proverb, that "the friends of our friends are our friends," for a proverb is not an axiom. Since all the embarrassment here is due to the complication of the relations considered, I am going, as a preliminary, to examine two cases, which have some resemblance to the present case, but which, being borrowed from geometry, offer more simplicity.

There is, to begin with, this case. Let there be three polygons, P, P', P''. The side α of the polygon P is supposed equal to the side α' of the polygon P', and the side β of the polygon P is equal to the side β'' of the polygon P''. It is evident that the two polygons P and P', being such as might have a side in common, are in virtue thereof connected by a symmetric relation, and that the same kind of a relation exists also between the polygons P and P''. If the axiom of symmetry is true, Mr. Russell would say similar relations ought to exist between the polygons P' and P'', and since we know that such is not the case and that these polygons are by no necessity such as must have any side in common, it would seem. that the axiom of symmetry is here at fault. The truth is, that it is not applicable. The axiom supposes only *three* terms or two *conjoint* relations, and in the case considered, there are four terms, α, α', β, β', and two *disjoint* relations, $\alpha = \alpha'$ and $\beta = \beta'$. In order for the axiom to be applicable, it would be necessary that the two relations should have a common term, which can only obtain if, say, $\alpha = \beta$, and then we would find ourselves brought back to an application of the axiom of Euclid with regard to equality.

We see, therefore, that before we can apply the axiom of symmetry to the relations subsisting between a set of component terms, it is necessary, in the first place, to ascertain that one of the several terms, which enter directly into those relations as essential, should be identically the same for one and another of the relations given. In any other case we would be in error in considering the relations as conjoint. We can only say that the axiom is true of simple determinate terms which cannot be decomposed without disturbing the relations that connect them.

The second case is also borrowed from geometry. Two symmetric *figures* in virtue of that very symmetry, are certainly connected by a symmetric relation, and yet we know that two *figures* that are symmetric with a third figure are not symmetric with each other.

If the axiom of symmetry still appears faulty, it is not because the relations considered are disjoint; it is because it is applicable only to simple relations that subsist between two determinate terms, while geometric symmetry is a relation involving three determinate terms, namely, the two figures considered and the point, axis, or plane, of symmetry.

We can now, by way of analogy, attack the complex case cited by Mr. Russell, and show why it is that the axiom of symmetry is not applicable. And in the first place, a person, a being, is not a unity, objectively considered, and as for moral personality, it is a very complex whole of various manifestations; subjectively, no more is the unity realised, and the researches upon hysteria have conclusively proved the theory of Hume that every mind is a loose or a more or less compact tissue of sensations, ideas, and emotions, sometimes even broken up and dispersed, if not even a tissue of sensations and simple relations between sensations. So from two points of view we may compare beings to sorts of moral polyhedrons, situated in a space of an indefinite number of dimensions, and formed of a quasi-indefinite number of faces in process of perpetual deformation.

Thus, relations of mutual friendship correspond to the possible fact that in the midst of these incessant transformations there is frequently between the two moral polyhedrons considered certain determinate faces in common. We can well understand by this figure that a mutual friendship does not directly connect two beings, but only certain of their manifestations, or certain of their hypothetical states of consciousness. Friendship connects persons in this only, that their manifestations or their states of consciousness make up an integral part of their personality. Frequently the manifestations α, under which appeared in C the friendship which connected him to A, are different from the manifestations β, which connected him to B. The faces common to the moral polyhedrons A and C are not, then, the same as the faces which are common to the polyhedrons B and C, and hence the polyhedrons A and B have not of necessity any faces in common, and the beings A and B are not of necessity mutual friends. But, as I desire to show, this does not impugn the axiom of symmetry. The conditions that it supposes are not fulfilled.

In the second place, in order to make definite any special relation of mutual friendship, it will not suffice merely to point out the two determinate persons connected by it, it is necessary also to show its *determinate* foundation, which is almost always a community of tastes, some common passions, some moral trait. In short, mutual friendship is a relation of three or more determinate terms. It is a sort of geometric symmetry, the quality in common here playing the rôle of an axis of symmetry. There exists also an infinity of kinds of relations of friendship, and the axiom of symmetry cannot rationally be applied to a case where the two relations given are of different sorts. For example, William, who is fond of hunting and fishing, has for friends, John, who is fond only of hunting, and James, who is fond only of fishing ; but John, who is afraid of rheumatism, and James, who dreads the fatigue of walking, have not by reason of these connexions with William any cause to connect themselves with one another.

In the third place (and really I find myself led nearly to the making of a scientific course of ethics) in the relations of moral and social order large account is to be taken of the complications which oppose the realisation of the possible relations.

We know that in mechanics, in chemistry, in physics, such complications intervene in a great number of cases and falsify the results that we would have the right to infer if the bodies were always able to obey the mechanical, thermal, elastic, or electric forces that influence them. In the dynamics of persons the same is the case, but the complications are besides frequent, complex, and extended. Liberty, that is to say considering merely the mental complications that restrain activity, is as little to be realised as is the absolute zero of temperature or of electric tension. Tyranny and the immorality of governments, the conflicting appetences of individuals, the contradictory passions of the individual himself are so many complications which conspire to restrain the human activity. It is only upon rare and transient occasions that the manifestations of any person are conditioned and governed by his own proper activity independently of the excitations or the restraints due to alien activities. We can sometimes observe such occasions but we can never produce them in their full estate, and thus many of the particular relations that are possible escape our notice.

All which being duly considered, that which we observe, that we call a relation between persons, is not the relation itself, it is its realisation, that is to say, the effective repetition of the facts upon which it is founded. Or a true relation, or at least the true notion of a relation, does not consist in the facts but in the prevision of the facts, those facts which emerge if A and B are brought together under certain determinate conditions, which conditions are, however, seldom realisable. Hence this conclusion follows, that in the study of the laws of moral science and of sociology that which is necessary to be considered if we wish to discover the actual laws we are seeking, are not the too complex facts of habitual observation but the simple facts which emerge if the conditions of solidarity are satisfied, and if the interior complications of the system considered are neglected. It is in this sense that we must understand the observation of Stuart Mill, that the sciences in question can be treated only by means of deduction. It is in this way that we are able and that we ought to regard those relations that while inaccessible to observation are, however, the real elements of the relations that are actually perceived.

For example, the *community* of tastes, the foundation of their mutual friendship is the same between William and John as between William and James, still if the circumstances are such that John and James do not happen to meet, or if there exists between the two an enmity arising from the opposition of their interests, it is evident that John and James may never become friends, or, in the latter case, may continue to dislike one another cordially, while at the same time loving William the common friend none the less. Nevertheless, we should by application of the axiom of symmetry infer a relation in the logical sense of the word between John and James. That axiom signifies, indeed, simply that certain manifestations occur if such and such conditions are satisfied. It does not signify that those manifestations actually occur.

To resume, I think I have shown that the axiom of symmetry, which I am not alone in enouncing, but which has often been invoked under various forms notably by DeMorgan ("Memoir," IV), is not undermined by the examples that Mr. Russell offers to me. But in order for it to be applicable it is necessary that the following conditions should be fulfilled.

The relations given must be two in number, neither of which has more than two entirely determinate terms opposed in sense to each other. The relations must be symmetric and possess a common term. If this common term is complex the two relations must refer essentially and exclusively to the same determinate common element.

All this can be summed up briefly by saying that the terms of the axiom are distinct, irreducible, and three in number, oppositely arranged in pairs by the same sort of relations independent of *sense*.

I may as well add that the axiom of symmetry is only a particular case of an axiom still more general which I have called the "axiom of the three senses," and which I have specially examined in a paper upon the "Le sens de l'inégalité" (*Revue Philosophique*, No. 197).

As regards this paper Mr. Benj. Ives Gilman has, with a luxury of symbols to my mind perfectly useless, taken the question up again in an article (*Mind*, New Series, No. 4) of which *The Monist* of January, 1893, contains a brief analysis. I am glad on this occasion to join in testifying that the fundamental principles which enable us to simplify and unite our objective cognitions are beginning to win for themselves a recognised place in the sphere of the philosophic studies.

II. SPENCER'S PRINCIPLE.

It is an error, in my judgment, for Mr. Russell to give the name of Mr. Spencer to the axiom of symmetry. Mr. Spencer has enounced a principle which is as follows: "Two things which have a definite relation to a third thing, have a definite relation to one another." But this principle of Mr. Spencer is distinct from the axiom of symmetry. It is completely separated therefrom by its more general object, by the nature of its conclusion. and by its subjective origin.

By its object.—The principle of Mr. Spencer refers to any two conjoint relations; the axiom of symmetry refers to two symmetric relations exclusively. The principle of Mr. Spencer is unqualifiedly general; the axiom of symmetry touches only a particular case.

By the nature of its conclusion.—It follows from this first point of distinction that before applying the axiom of symmetry to any particular case, it will be advisable to apply to it the principle of Mr. Spencer. This principle then makes known to us that between two things connected to a third thing by a symmetric relation there exists a certain definite relation, but it makes known nothing further. It does not make known the nature of that relation, which is completely definite the moment that its relational elements are given. The axiom of symmetry adds another conclusion to this first one because it tells us that between the two things under consideration there exists, beyond the new relation that has just been spoken of and defined, a symmetric relation which is *of the same nature* as are the symmetric relations given. The first relation inferred is made definite *by* these, but the second relation is certified as the *same* as these. In practice these two kinds of relations become confused together or rather the first one escapes notice so that the same has not received any special name.

By its subjective origin.—The axiom of symmetry expresses a fact of experience, a conclusion that is not contained in the premises, and of which the negation can without difficulty be conceived. The proof of this is the very objection of Mr. Russell, who not only conceives this negation but affirms the reality of it. The principle of Mr. Spencer is, on the contrary, a condition of conception. We cannot even conceive that it may be not true. In the language of Kant we might say that the principle of Mr. Spencer is analytic and that the axiom of symmetry is synthetic: I prefer, however, to say that the former is a law of thought and that the latter is a law of nature.*

As a corollary the principle of Mr. Spencer borrows nothing from the axiom of symmetry and that axiom may be found defective without compromising the exactitude of the principle.

III. MY THEORY OF RELATIONS.

The true point of departure of the theory that I have given of the nature of concepts is not and cannot be the axiom of symmetry, since that is a simple objective verity which does not directly concern the mode of the development of cognition. This point of departure is the principle of Mr. Spencer or rather its reciprocal about which Mr. Spencer has not spoken.

Generalising his principle, to begin with, we may say that every complex of relations constitutes a new concept. Then I go still further and reversing the conclusion I have given this first and fundamental principle.

Every concept, every relation is made up of a complex of relations.

This principle in practice scarcely advances us to the point of view of the analysis of ideas. If the relations which enter into a concept resolve only in functions of the concept itself, if in analysing any given notion we must after a course more or less extensive return again to the same notion we will thus only go round and round in a circle. But I have added the second fundamental principle which overcomes the difficulty.

The series of concepts established by the first principle is a continuous one and it has a double SENSE—*unique and determinate.*

There is in point of fact an ascending *sense* from the particular to the general and a descending *sense* from the general to the particular, and in each sense the same concept is once only to be encountered. Every concept occupies in the general series a unique and determinate position. We are thus able to say that every concept is derived from concepts that are more simple and more directly known, and itself gives birth to concepts that are less simple and less directly known. As a corollary we must take notice that every concept is subject to a single, logical definition, a truth too often overlooked, thanks to the perpetual metaphysical confusion between notions and things, between cognition and its objects.

---

* Between the principle and the axiom there is the same kind of an opposition as there is between a notion and a law, an opposition upon which I have insisted in referring to the article of Mr. Hobhouse, noticed by Mr. Russell (*Mind*, New Series, No. II).

I have ended by this third principle, not less fundamental, which has been known ever since the time of Aristotle, and which DeMorgan was one of the first to restate in a scientific form, that is the principle of relativity, namely :

Every concept supposes another one which is opposite (in nature) and single; each one of the two contrary concepts being defined by the negation of the other in the relational complex constitutive of a more general concept, one relation being affirmed by the opposite concept, and inversely. That is to say, to borrow a common image, that the tree which represents the development of cognition and whose roots are our states of consciousness, does not ramify in whorls, but only in dichotomies.

These three principles of logic or fundamental laws of thought (entirely distinct from the scholastic tautologies known under the same name) the principle of constructivity, the principle of logical evolution, and the principle of relativity* cover the entire subject.

It is convenient to separate them from the subsidiary conditions, those of coexistence and of solidarity, whose rôle is simply to define what is to be understood by a complex of relations. In regard to what I have called the condition of abstraction (or the principle of indetermination) which appears to frighten Mr. Russell, it is only a way of expressing the fact that in each concept there is a determinate degree of generality, a degree which conditions the indispensable connexion between the concept and the reality of the facts that the concept supposes. This condition of abstraction is the conceptual counterpart of the law abstractly enounced by Stuart Mill under the name of the law of the uniformity of nature. It is needless to stumble over the meaning of this condition. It simply means that in order to be general, every cognition must be somewhat indeterminate, which indetermination does not, as Mr. Russell thinks, reside in the relational elements, those being perfectly determinate, but resides in some of the terms, in the last analysis in some of the things or states of consciousness apart from the concepts. The concrete examples which I have given forbid any mistake in that respect.

I come now to the most important of the observations which Mr. Russell has formulated. At the very beginning of my paper I stated this preliminary question, What is a relation and what is a concept? Mr. Russell objects that it is not sufficient to show that a relation is a composite of other relations more directly known, that such is not a satisfactory answer to the question.

To answer the question there are two points of view at which one can place himself, the logical point of view and that of pure psychology.

The second point of view is that of Mr. Russell. The question is then to ascertain not what is *a* relation but what is relation, what may be that which constitutes the subjective element, directly apparent, which causes a determinate assemblage of states of consciousness to have a relative character.

---

* This would be better named the principle of duality, the word relativity having been put to so many uses that it has finished by losing all precise meaning.

This question I have certainly not answered, and I have not answered it because I have nothing to do with it. I took good care from the very beginning of my papers,* even before stating any question whatever, to mark out my range and to reduce it to that of pure logic, merely.

I took as data all cognitions that were established and indisputable, all the principles and all the notions that we study in psychology, all those of the exterior world, those of matter, of time, of space, and even the notion of relation itself. In fine to avoid every encroachment upon psychology I even forbade myself from choosing any of my concrete examples of quantity from geometry or kinematics. I thus placed myself at the logical point of view exclusively, and I hold it for certain that I have satisfied the exigences of that science by showing that the concepts of scientific order are composed of less complex elements which are reducible to differences of the divers states of consciousness, by pointing out the order and the conditions of the derivation of concepts, in a word, by describing all the conditions for the passage from the known to the unknown independently of the question of the primary origin of the known.

There is, it is true, in the theory which I maintain, an important, though voluntary gap, a gap, alas! that certain logicians would find regrettable. I have not studied concepts under their verbal forms of expression, in their relations with language. This is because in accordance with the great philosopher, Mr. Spencer, of whose labors I avail myself, I consider that all questions that relate merely to the ways of expressing thought are foreign to the domain of pure logic. Language, even philosophic language, is an affair of the grammar and the dictionary. The questions as to the distinguishing of all the significations of the copula "is," as to all the forms of the syllogism, as to all the "quantifications" of propositions, belong only indirectly to logic. Besides, this mode of dealing with reasoning, the triumph of scholasticism, has been completely treated by a very great number of authors. They have said all that there is to be said upon the subject, and peevish minds might perhaps add that they have said much more than there was to be said. Our usual language formed by the necessities of life has been adjudged insufficient and symbols have been substituted for it. "Reasoning machines" have even been invented. As for myself, without going so far as to regard as certain that these "improvements" have been rather of a nature to discredit logic than to put it upon a scientific basis, I nevertheless hold that they ought to be excluded from the region of pure logic. I believe that no progress in the laws of the latter can be attained by any combination of symbols however learned they may be, but only by a study perhaps arduous but necessary, of the laws and notions that the sciences offer to us.

---

* See especially pages 65 and 66 of No. 181 of the *Revue Philosophique*, also note 2 on page 117 and pages 132 and 133 of No. 188.

Rightly understood, however, I do not dream of excluding from logic, as I regard it, the questions relating to its proper nomenclature. It is plain that it would be useful to unify this, which would be only to avoid divergences of language analogous to those that have arisen between Mr. Russell and myself.

"Relations," observes Mr. Russell, "are attributive predicates of terms." For instance, between father and son Mr. Russell sees two distinct relations, that of the father to the son and that of the son to the father. Conforming myself, I believe to the ordinary language as well as to the language of philosophy, I see only in that case a single relation, and what Mr. Russell calls a simple relation is only the sense in which one regards a relation, or if one pleases the relative place of a term in a relation or in a complex of relations, that is to say that which is ordinarily called an attribute of a term. "Dual relations" are what I call a relation and "relationship," what I call the foundation. Very certainly it would be preferable to have a common terminology, but it is not, however, necessary to infer from the divergence of language as my honorable opponent tends to do, a difference of doctrine.

There is, nevertheless, a matter in which I am not able to follow Mr. Russell in his reflexions upon the theory of relations, because this divergence of language then involves not merely a matter of words, but a matter of ideas. After having chosen as an example of relations, a complex of three determinate terms, that of the donation of a determinate thing, Mr. Russell passes to the consideration of the relation of the mutual attraction between bodies and would there find still three determinate terms comprising besides the two bodies between which the attractions exist a certain "mediating" term which he calls the attraction of gravitation. It is sure that this way of regarding relations has no correspondence with the theory which I maintain and of which the very foundation is the exclusion of every metaphysical entity, an exclusion made by George Berkeley and by David Hume, and for which there exists no reason whatever to return.

IV. MATHEMATICAL EQUALITY.

Mr. Russell is mistaken as to the object of the second part of my paper, which treats of mathematical equality, and his mistake appears to me scarcely explicable when I consider the care I took to multiply the concrete examples.

"There is," says Mr. Russell, "numerical equality upon which the equality in service in numeric algebra is founded, and there is geometric equality, the equality of vectors, etc.," all "different from one another. M. Mouret seems to have numeric equality only, in view." Now, in order to give an exact account of my doctrine, this assertion should be just turned about. Of all the conceivable kinds of equality, the single one that I did not meddle with was precisely numeric equality. The particular equalities which I analysed in detail were the equality of mass, of stress, of temperature, of quantity of heat, etc., independently of all questions of measure, because as I have said and as I am going to repeat, measurement supposes equality. Indeed my paper might better have borne the title "Mathematical Equivalence," and if I had recourse to the word "equality" it was only to conform myself to the ordinary language, the word equivalence being reserved for very special cases.

There exists at least in France a numerous class of authors, mostly composed of mathematicians, who consider mathematics as the science of combinations, having for points of departure certain conventions made with numbers, independent of reality in general and of the physical magnitudes in particular.*

It is partly by way of protest against these superficial doctrines, against these philosophical paupers that I have occupied myself in analysing the fundamental notions of mathematics, and my design on this behalf has been to bring into view the truth that under the symbols and notions of mathematics there are notions relatively concrete. If geometry, mechanics, or the physical sciences had not existed there would have been no mathematics possible. As I have said elsewhere, a symbol is such only on condition of symbolising something; otherwise it is nonsense. Numerical equality is only an abstraction symbolised. I had to show that which is concealed underneath the equivalence of masses, of forces, of vectors, etc. This I believe I have done, but I have not preoccupied myself with numerical equality itself. That is a matter I have left entirely aside.

On the subject of the rôle played by equality, numerical or not, in mathematics, Mr. Russell has made a remark, which I must examine, because it tends to nothing less than to suppress the mathematical relativity. I have stated as a principle, that the relation of equality is fundamental and that the very existence of mathematics is bound up in the existence of that relation. Mr. Russell observes that there exists, or that there might exist, a branch of algebraic or numeric analysis having for its sole avail the relation of inequality. He adds that the relation of equality is itself a composite of the two relations "not less than" and "not more than," that is to say, derived from inequality. He concludes that a mathematics can be formed without the use of the relation of equality.

All these assertions are entirely correct, but they do not infringe upon my theory. I said, and I give again my own expressions, "mathematics would not exist, if we "did not have the *notion* of mathematical equality. All the functions (mathematical relations) *involve* this relation." I did not say that every mathematical relation necessarily *comprises* this relation. That would not be exact. What I persist in maintaining is that mathematics essentially supposes equality; in precise terms, that we cannot conceive of any mathematical relation without being constrained, as a preliminary, to conceive relation of equality. The reason which I invoke is the mathematical relativity.

My opponent remarks with truth, that the relation of equality may be considered as the negation of two relations of inequality of opposite senses,—this is precisely what I explained with detail in my paper upon " Le sens de l'inégalité,"—but it ought to be added that inequality itself can also be considered as the negation of equality. At bottom, equality and inequality are two correlative notions logically equivalent to one another. It is as easy to dispense with the one as with the other, nor can we think of the one without thinking also of the other. The thought, or rather the cognition, of these two relations is unique, but it has two faces, two contraries, as Hegel said ; two contraries that mutually support and determine one another in the same manner and for the same reason that the idea of one half of an apple implies the idea of the other half, even though that other half may be swallowed and digested.

---

* Not all the mathematicians, however, partake of this opinion. Among others, Dr. Paul Du Bois-Reymond, in Germany, M. M. Sorel, Couturat, etc., in France. See especially on this subject the excellent remarks of M. Couturat in the first number of the new *Revue de Metaphysique et de Morale*, p. 79.

It appears to me, then, not rash at all to maintain that without the notion of equality there would be no mathematics, and this for exactly the same reason that the same would not exist without the notion of inequality.

I must now excuse myself for having been so lengthy, but truly the occasion was a tempting one, for Mr. Russell presented his objections with such clearness and sincerity as to give me solid ground for some, perhaps indispensable, elucidations.

George Mouret.